of determining the directions in which valency acts. It is well known that this is a problem which has attracted much attention of late years in the case of nitrogen.

We are now studying the salts of tervalent metals generally, in order to ascertain if the method be applicable to all such elements: it is proposed also to apply the method to the elements of the nitrogen group.

In conclusion, we cannot refrain from calling attention explicitly to the proof that is again given of the service crystallography can render to chemistry and of the value of the Barlow-Pope theory as a means of correlating external form with internal molecular structure.

On the Convergence of Certain Series Involving the Fourier Constants of a Function.

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§1. In a paper "On the Fourier Constants of a Function," published in the 'Proceedings' of this Society,* I showed how certain theorems, previously given by myself, might be employed to obtain formulæ for the sums of certain series involving the Fourier constants of a function. In the case in which the function is bounded, it was only proved that the formulæ† hold whenever the series converge, or, more generally, when the summation is performed in the Cesàro manner (index unity). In a more recent paper, also published in the 'Proceedings' of this Society,‡ "On a Mode of Generating Fourier Series," I showed incidentally that the formulæ are still applicable when the function has every power of index less than $1+p$ summable, provided only the index q, which occurs in the formulæ in question, is greater than $1/(1+p)$. Here again the theorem, as stated, contained the restriction that the summation was to be made in the Cesàro way. The main object of the papers in question was, in fact, to explain how certain methods might be employed, and these methods were in themselves inadequate for the purpose of removing the restriction. The theorems used involve the general theory of the integration of the Fourier series of a function term-by-term, when multiplied by another function. The coefficient n^{-q} in the series $\sum_{n=1} n^{-q}a_n$ and $\sum_{n=1} n^{-q}b_n$ considered, is, however, itself the typical Fourier constant both of an odd and an even function, which may be expected to possess special properties bearing on the matter in hand. A careful scrutiny of these properties has accordingly enabled me to take the step of removing the restriction above explained.

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* 1911, A, vol. 85, pp. 14—24.

† *Loc. cit.*, § 4.

‡ 1911, A, vol. 85, p. 422.

The former of the main results, above stated, may be otherwise expressed by saying that, though the Fourier series of a bounded function need not converge, even if the function be continuous, it, and its allied series, will be made to converge, by dividing its coefficients by any power, however small, of the index denoting their respective places in the series. This affords a convenient necessary test that a given Fourier series is the Fourier series of a bounded function. In the same way we have a corresponding necessary condition that a Fourier series should have a function whose $(1+p)$th power is summable for associated function.

It is worth noticing that in all these statements the function whose Fourier series is considered need not be homogeneous in character. They still hold if the function be only summable, except in a fixed interval containing the origin.

The paper terminates with a comparison of the result of the present paper regarding a function whose $(1+p)$th power is summable, and one deducible from a theorem given in my paper, " On the Mode of Generating Fourier Series." From the theorem in question it, in fact, may be shown that the series whose general terms are $n^{-\frac{1}{2}(q+1)}a_n$ and $n^{-\frac{1}{2}(q+1)}b_n$ converge absolutely, when $q > (p+1)^{-1}$.

It should finally be remarked that the interesting question arises as to how far the coefficient n^{-q} may be replaced in the theorems of the present paper and its predecessors by a coefficient of a more general nature. This question is too large a one to be dealt with in the present communication.

§ 2. We begin with the following general theorem :—

Theorem.—*The succession of the partial summations of the series*

$$\sum_{n=1} n^{-q} \sin nx, \qquad (0 \leq x \leq \pi),$$

is bounded below, whatever fixed positive value be ascribed to q. *The same is true of the series* $\sum_{n=1} q_n \sin nx$, *where* $q_1, q_2, \ldots$ *is any monotone descending sequence with zero as limit.*

Let s_n denote the partial summation of the latter series, which includes the former as a special case. Then

$$2s_n(x)\sin x = \sum_{m=1}^{m=n} q_m [\cos(m-1)x - \cos(m+1)x] = \sum_{m=1}^{m=n} q_m (t_{m+1} - t_{m-1})$$

where $$t_m = 1-\cos mx = 2\sin^2 \frac{m}{2} x.$$

Hence

$$2s_n(x)\sin x = -q_2 t_1 + (q_1-q_3)\,t_2 + (q_2-q_4)\,t_3 + \dots \\ + (q_{n-2}-q_n)\,t_{n-1} + q_{n-1}t_n + q_n t_{n+1}.$$

Here every term except the first is positive, since $t_m \geq 0$, and the q's form a positive monotone decreasing sequence. Thus

$$s_n(x)\sin x \geq -q_2 \sin^2 \tfrac{1}{2}x,$$

and, therefore, in the interval $(0 \leq x \leq \tfrac{1}{2}\pi)$.

$$s_n(x) \geq -\tfrac{1}{2} q_2 \tan \tfrac{1}{2}x \geq -\tfrac{1}{2} q_2. \tag{1}$$

Again

$$2s_n(x)\sin \tfrac{1}{2}x = \sum_{m=1}^{m=n} q_m \left[\cos\left(m-\tfrac{1}{2}\right)x - \cos\left(m+\tfrac{1}{2}\right)x\right] = \sum_{m=1}^{m=n} q_m\,(u_{m+1}-u_m),$$

where $$u_m = 1-\cos\left(m-\tfrac{1}{2}\right)x.$$

Thus $$2s_n(x)\sin \tfrac{1}{2}x = -q_1u_1 + (q_1-q_2)\,u_2 + \dots + (q_{n-1}-q_n)\,u_n + q_n u_{n+1}. \tag{2}$$

Since each term on the right after the first is positive

$$2s_n(x)\sin \tfrac{1}{2}x \geq -q_1\, 2\sin^2 \tfrac{1}{4}x,$$

and therefore, in the interval $(\tfrac{1}{2}\pi < x \leq \pi)$,

$$s_n(x) \geq -\tfrac{1}{2} q_1 \tan \tfrac{1}{4}x \leq -\tfrac{1}{2} q_1. \tag{3}$$

From (1) and (3) it follows that, in the interval $(0 \leq x \leq \pi)$, since $q_1 \geq q_2$, $s_n(x) \geq -\tfrac{1}{2} q_1$, which proves the theorem.

Cor.—*The partial summations of the series*

$$q_1 \sin x + q_3 \sin 3x + \dots + q_{2n-1}\sin(2n-1)x$$

are all positive, provided $q_1, q_3, \dots$ *form a monotone sequence with zero as limit.*

The proof then stops at (1), which holds for $0 \leq x \leq \pi$.

It is easily seen from (2) that the partial summations are also bounded above in the right-hand half of the interval. This is, however, obvious from the well-known fact that series of the form $\Sigma q_n \sin nx$, as well as those of the form $\Sigma q_n \cos nx$, when the q's form a monotone descending sequence with zero as limit, converge uniformly in every closed interval (e, π) not containing the origin. This follows immediately by the use of Abel's Lemma.

Since the series is a sine series, it immediately follows that, *in the interval* $(-\pi \leq x \leq 0)$, *the partial summations of the series are bounded above, and are negative if* $q_2, q_4, \dots,$ *are zero.*

§ 3. The preceding result permits us to prove the following theorem:—

Theorem.—*If* $f(x)$ *is any bounded function, whose Fourier sine constants are* $b_1, b_2, \dots$ *then the series*

$$\sum_{n=1} n^{-q} b_n, \qquad (0 < q)$$

converges to

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, g(x)\, dx,$$

where $g(x)$ *is the sum of the Fourier series* $\sum\limits_{n=1} n^{-q}\sin nx$.

Moreover, the latter series, after multiplying each term by $f(x)$, *may be integrated term-by-term over any interval or set of points.*

It is evidently only necessary to prove the latter statement, from which the former statement immediately follows, choosing for the interval of integration the whole interval $(-\pi, \pi)$.

Since the series $\sum\limits_{n=1} n^{-q}\sin nx$ is an ordinary Fourier series, the integrated series converges to a Lebesgue integral,[*] whose integrand is the function corresponding to the Fourier series, that is $g(x)$, since the Fourier series converges everywhere. In other words, the Fourier series in question is capable of term-by-term integration.

Again, by the preceding theorem, the partial summations $s_n(x)$ of this Fourier series are bounded below in the closed interval $(0, \pi)$, and bounded above in the closed interval $(-\pi, 0)$.

Now Vitali[†] has proved that, if the partial summations of a series of functions are bounded either below or above in any interval, the necessary and sufficient conditions that the series should be capable of term-by-term integration are (*a*) that the series should converge,[‡] and (*b*) that the double limit

$$\operatorname*{Lt}_{E\to 0,\, n\to\infty} \int_E s_n(x)\, dx = 0.$$

Therefore, $s_n(x)$ denoting the partial summation of the Fourier series $\sum\limits_{n=1} n^{-q}\sin nx$, the preceding double limit is zero, and therefore, $f(x)$ being a bounded function,

$$\operatorname*{Lt}_{E\to 0,\, n\to\infty} \int_E f(x)\, s_n(x)\, dx = 0.$$

The series $\sum\limits_{n=1} f(x)\, n^{-q}\sin nx$ is therefore, by the theorem of Vitali already quoted, integrable term by term, since $f(x)\, s_n(x)$ converges to $f(x)\, g(x)$.

Moreover, by a further theorem of Vitali's,[§] this series is absolutely integrable, that is to say, the statement made in the enunciation of the present theorem is true.

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[*] See W. H. Young, "On the Conditions that a Trigonometrical Series should have the Fourier Form," 'Lond. Math. Soc. Proc.,' 1910, Series 2, vol. 9, p. 423.

[†] G. Vitali, "Sull' Integrazione per Serie," 'Rend. di Palermo,' 1907, vol. 23, pp. 137—155; see, in particular, § 9. The form of the theorem used above is that given to it in my paper "On Semi-integrals and Oscillating Successions of Functions," 'Lond. Math. Soc. Proc.,' 1910, Series 2, vol. 9, pp. 286—324; see §§ 23, 24, and 26.

[‡] An exceptional set of content zero of course makes no difference.

[§] *Loc. cit.*, §§ 1, 6, and 7. This theorem, as well as the other, is included in the articles quoted from "Semi-integrals."

This proves the theorem.

§ 4. To prove the corresponding statement for the coefficients of the cosine terms, we have to show that $\int_E s_n(x)\,dx$ vanishes, s_n denoting the n-th partial summation of $\sum\limits_{n=1} n^{-q} \cos nx$. For this purpose we shall require the following lemma :—

Lemma.—*The function* $\sin nx/(\sin x)^q$, *as well as* $\sin nx/x^q$, *satisfies inequalities of the form*

$$\tfrac{1}{2}An^q \leq \sin nx/(\sin x)^q \leq \tfrac{1}{2}Bn^q, \qquad (0 < q \leq 1;\ 0 < x),$$

where A and B depend only on q.

This is an immediate consequence of the fact that, since $\sin nx$ is periodic and x^q is monotone, the function $x^{-q} \sin nx$, that is $n^q [(nx)^{-q} \sin nx]$, is a bounded continuous function, whose greatest and least values are situated in the interval $(0, 2\pi)$.

§ 5. In dealing with the sine series we proved that the partial summations are all greater than a certain negative quantity on the right of the origin, and less than a certain positive quantity on the left. It was unnecessary to examine whether the partial summations were as a matter of fact positive, as is known* to be the case when q is unity. In dealing with the cosine series I shall again prove a result *ad hoc*, and shall not examine whether, as is the case when q is unity,† the partial summations are all greater than a certain negative finite quantity. It is sufficient in fact for our purpose to show that they are all greater than a certain negative summable function.

Theorem.—*The partial summations of the series*

$$1 + \sum_{n=1} n^{-q} \cos nx \qquad (0 < q \leq 1;\ 0 < x)$$

are all greater than Ax^{q-1}, *where* A *is a negative constant depending only on q.*

Writing $S_n(x)$ for the n-th partial summation of this series, we have

$$\begin{aligned} 2 \sin \tfrac{1}{2}x\, S_n(x) &= (1 - 1^{-q}) \sin \tfrac{1}{2}x + (1^{-q} - 2^{-q}) \sin \tfrac{3}{2}x + \dots \\ &\quad + [(n-1)^{-q} - n^{-q}] \sin \tfrac{1}{2}(2n-1)x + n^{-q} \sin \tfrac{1}{2}(2n+1)x \\ &= T_n(x) + n^{-q} \sin \tfrac{1}{2}(2n+1)x, \end{aligned}$$

where the partial summation $T_n(x)$ has been shown in § 2 to be positive, $(0 \leq x \leq \pi)$. Therefore, using the preceding lemma,

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* Dunham Jackson, "Ueber eine trigonometrische Summe," 'Rend. di Palermo,' 1912, vol. 32, pp. 257—263.

† See a forthcoming paper by myself, "On a Certain Series of Fourier," presented to the London Mathematical Society.

$$S_n(x)-\tfrac{1}{2}\operatorname{cosec}\tfrac{1}{2}x\,.\,T_n(x)=\tfrac{1}{2}n^{-q}\frac{\sin\frac{1}{2}(2n+1)x}{(\sin\frac{1}{2}x)^q}(\sin\tfrac{1}{2}x)^{q-1}$$

$$\geq\tfrac{1}{2}A\left(1+\frac{1}{2n}\right)^q(2\sin\tfrac{1}{2}x)^{q-1}>Ax^{q-1},$$

which proves the theorem, since $\frac{1}{2}\operatorname{cosec}\frac{1}{2}x\,.\,T_n(x)\geq 0$.

§ 6. We now proceed to apply the preceding result to the series whose general term is $n^{-q}a_n$, where a_n is the typical Fourier cosine constant of a bounded function.

Theorem.—*If $f(x)$ is any bounded function, whose Fourier cosine constants are $a_1, a_2 \ldots$, then the series*

$$\sum_{n=1} n^{-q}a_n, \qquad (0<q),$$

converges to

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,S(x)\,dx,$$

where $S(x)$ is the sum of the Fourier series $\sum\limits_{n=1} n^{-q}\cos nx$.

Moreover the latter series, after multiplying each term by $f(x)$, may be integrated term-by-term over any interval or set of points.

For, by the preceding theorem,

$$S_n(x)-Ax^{q-1}\geq 0,$$

and the integral of this function converges to $\int S(x)\,dx-Aq^{-1}x^q$. Therefore, as in § 3,

$$\operatorname*{Lt}_{E\to 0,\; n\to\infty}\int_E [S_n(x)-Ax^{q-1}]\,dx=0,$$

and therefore, since Ax^{q-1} is summable,

$$\operatorname*{Lt}_{E\to 0,\; n\to\infty}\int_E S_n(x)\,dx=0.$$

Hence, as in § 3, the required results follow.

§ 7. We now pass to the case in which $f(x)$ has its $(1+p)$th power summable. We may, if we please, use the lemma of § 4 once more, and write

$$\operatorname{cosec}\tfrac{1}{2}x\,T_n(x)\leq B\,\{2\operatorname{cosec}\tfrac{1}{2}x\}^{1-q_1}\sum_{r=1}^{r=n}\{(r-1)^{-q}-r^{-q}\}\,(r-\tfrac{1}{2})^{q_1},$$

where B is >0 and depends only on q_1. This partial summation is less than a finite quantity independent of n, provided $q_1<q$, since $[(r-1)^{-q}-r^{-q}]$ is of the order r^{-1-q}. Hence for such values of q_1

$$\operatorname{cosec}\tfrac{1}{2}x\,T_n(x)\leq Cx^{q_1-1},$$

and, therefore, for all values of $p_1<q_1$, that is for all values of $p_1<q$ $\operatorname{cosec}\frac{1}{2}x\,T_n(x)$ has its $1/(1-p_1)$th power summable; moreover,

$$\underset{E\to 0,\ n\to\infty}{\mathrm{Lt}} \int_E [\operatorname{cosec} \tfrac{1}{2}x\, T_n(x)]^{1/(1-p_1)}\, dx = 0. \qquad (0 < p_1 < q),$$

But since, by § 5, $Ax^{q-1} < S_n(x) - \frac{1}{2}\operatorname{cosec}\frac{1}{2}x \,.\, T_n(x) < Bx^{q-1}$,

$$\underset{E\to 0,\ n\to\infty}{\mathrm{Lt}} \int_E \left| S_n(x) - \tfrac{1}{2}\operatorname{cosec}\tfrac{1}{2}x \,.\, T_n(x)\right|^{1/(1-p_1)} dx = 0,$$

whence, since*

$$\left\{\int_E |S_n(x)|^{1/(1-p_1)}\, dx\right\}^{1-p_1} \leq \left\{\int_E |S_n(x) - \phi(x)|^{1/(1-p_1)}\, dx\right\}^{1-p_1}$$
$$+ \left\{\int_E |\phi(x)|^{1/(1-p_1)}\, dx\right\}^{1-p_1},$$

we have
$$\underset{E\to 0,\ n\to\infty}{\mathrm{Lt}} \int_E |S_n(x)|^{1/(1-p_1)}\, dx = 0.$$

Writing $\dfrac{1}{1-p_1} = 1 + \dfrac{1}{p}$, we have $p_1 = \dfrac{1}{1+p}$.

Hence, finally, *if* $S_n(x)$ *denote the partial summation of the series,*

$$1 + \sum_{n=1} n^{-q} \cos nx,$$

then for all positive values of p such that $(1+p)^{-1} < q$, *we have*

$$\underset{E\to 0,\ n\to\infty}{\mathrm{Lt}} \int_E S_n(x)^{1+1/p}\, dx = 0.$$

§ 8. From this last result it at once follows† that, *if* $f(x)$ *is any function whose* $(1+p)$*th power is summable, and* $S_n(x)$ *has the meaning of the preceding statement,*

$$\underset{E\to 0,\ n\to\infty}{\mathrm{Lt}} \int_E S_n(x) f(x)\, dx = 0,$$

provided only that $(1+p)^{-1} < q$.

Hence, under these circumstances, the series

$$\sum_{n=1} n^{-q} a_n$$

converges, a_n *being the coefficient of* cos nx *in the Fourier series of* $f(x)$.

By a method similar to that of the preceding article we may establish a corresponding result for the cosine coefficients of the Fourier series of function $f(x)$ whose $(1+p)$th power is summable, but I shall not give the proof. Indeed, my chief object in writing out the proof just given is to further illustrate the usefulness of the lemma of § 4. If we make use of a result already obtained in a previous paper,* namely that the Fourier series obtained by the process in question

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* F. Riesz, "Untersuchungen über Systeme integrierbarer Funktionen," § 2, 'Math. Ann.,' 1910, vol. 69, p. 456.

† W. H. Young, "On Successions of Integrals and Fourier Series," § 6 Cor. and § 4 Cor., 'Lond. Math. Soc. Proc.,' 1912, Series 2, vol. 11, pp. 57 and 52. We only have to use the Schwarz-Riesz inequality.

$$\Sigma\, n^{-q}\,(a_n \cos nx + b_n \sin nx),$$

and its allied series

$$\Sigma\, n^{-q}\,(b_n \cos nx - a_n \sin nx),$$

have continuous functions for their associated functions, we may dispense with further proof. If q, in fact, be any quantity greater than $1/(p+1)$, we may perform the process of dividing by n^q in two stages. We may first divide by n^q, where $1/(1+p) \leq q' < q$; we thus get Fourier series of continuous, and therefore bounded, functions. Then, dividing by $n^{q-q'}$, it follows by §§ 3 and 6 that *the series* $\Sigma n^{-q} a_n$ *and* $\Sigma n^{-q} b_n$ *converge, when* a_n *and* b_n *are the Fourier constants of a function whose* $(1+p)$*th power is summable, provided* $1/(1+p) < q$.

§ 9. It will be noticed that, *in the case of a function whose* $(1+p)$*th power is summable,* what we have proved is that, *if* $q > (p+1)^{-1}$, *the series whose general terms are* $n^{-q} a_n$ and $n^{-q} b_n$ *necessarily converge;* we have not proved that the convergence is absolute. In this connection it may be well to point out that one of the simple consequences of the last result obtained in my paper "On a Mode of Generating Fourier Series" is that *the series whose general terms are respectively*

$$n^{-\frac{1}{2}(q+1)} a_n \quad \textit{and} \quad n^{-\frac{1}{2}(q+1)} b_n$$

converge absolutely under the same circumstances. This follows, utilising simultaneously the convergence of the two series $n^{-q} a_n^2$ and n^{-1-e}.

§ 10. Another interesting consequence of the same theorem may also be mentioned. *If* a_n *and* b_n *now denote the Fourier constants of the integral of a function whose* $(1+p)$*th power is summable, we can assert that the series whose general terms are* $|a_n|^r$ *and* $|b_n|^r$ *converge, if* r *has any value* $> (2p+2)/(3p+2)$.

In particular, however small p may be, the corresponding Fourier series and allied series both converge uniformly and absolutely.

To prove this we have merely to employ the inequality

$$(1+k)\, uv \leq u^{1+k} + kv^{1+1/k},$$

making use of the converging series $\Sigma n^{2-q} a_n^2$ and Σn^{-1-e}, where $q > (p+1)^{-1}$, and writing u^{1+k} and $v^{1+1/k}$ for the general terms of these series, k being so chosen that $2-q = k(1+e)$.

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* W. H. Young, "On a Mode of Generating Fourier Series," 'Roy. Soc. Proc.,' 1911, A, vol. 85, p. 421.